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1989 J. Phys. A: Math. Gen. 22 4271

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J. Phys. A: Math. Gen. **22** (1989) 4271–4280. Printed in the UK

The U(16) algebraic lattice

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Received 18 January 1989

Abstract. We detail the full subalgebra lattice of U(16), the algebra encountered in the bosonic quantisation of asymmetric shapes described by $\lambda^{\mathrm{P}} = 0^+$, 1^-, 2^+ and 3^- multipoles. All subalgebras that can be generated from tensors built from 0^+, 1^-, 2^+ and 3^- bosons and that have the proper O(3) content are identified from elementary representation theory. Of the 165 dynamical symmetry limits, we present seven dynamical symmetry limits and suggest that they reflect the relevant symmetries of this model.

1. Introduction

The U(16) algebra arises from the bosonic quantisation of a classical shape in terms of $\lambda^{\mathrm{P}} = 0^+, 1^-, 2^+$ and 3^- multipoles. It was initially introduced to describe features of nuclei that are reflection asymmetric in the ground state (Engel and Iachello 1985) and has been discussed more recently in a bag-like description of baryonic spectra (Iachello 1988). To date, the application of U(16) Hamiltonians has been limited to the phenomenological description of low-energy nuclear collective excitations. In the past, such interacting boson models (IBM) have been quite successful, largely due to the algebraic structure and dynamical symmetry properties of these models (Iachello and Arima 1987). However, attention has been primarily focused on positive parity collective excitations generated by s and d bosons ($\lambda^{\mathrm{P}} = 0^+, 2^+$, respectively). This was due, in part, to the inability to describe electric dipole transition probabilities in nuclei using octupole bosons ($\lambda^{\mathrm{P}} = 3^-$; f bosons) coupled to the positive parity states generated by s and d bosons (the sdf IBM). Recently there has been a great deal of experimental activity measuring electromagnetic electric dipole transition properties of positive and negative parity states in nuclei where there is suspected ground-state octupole deformation. In order to attempt a collective model description of the electric dipole transition properties and study the implications of octupole deformation in the IBM framework, an interacting boson model containing s, p, d and f bosons has been introduced (the spdf IBM) (Han *et al* 1985, Engel and Iachello 1987). The one- and two-body interactions constructed from s, p, d and f bosons close under the U(16) Lie algebra. Phenomenological Hamiltonians constructed from U(16) have been applied to the neutron rich rare-earth nuclei (Kusnezov and Iachello 1988) and the light actinide nuclei (Engel and Iachello 1987, Otsuka and Sugita 1988) with some success. It is our purpose here to classify completely the limiting symmetries of this model. The sdf IBM will be seen to be only a small contribution to the large number of dynamical symmetries encountered in the spdf IBM, which can then be understood to account for the success of the latter model.

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The U(16) algebra and subalgebras are also of interest in the study of bose–fermi symmetries in odd–odd nuclei. Particularly, U(16) arises as the symmetry group when two quasiparticles corresponding to neutron and proton degrees of freedom are given $j = \frac{3}{2}$ configurations (Hübsch and Paar 1984). Upon constructing generators that mix the protons and neutrons, the U(16) algebra is obtained as a maximal symmetry. The subalgebra U(4) $\oplus$ U(4) then emerges as a natural decomposition of U(16) into separate proton and neutron U(4) subalgebras, one for each $j = \frac{3}{2}$ configuration. The similarity in the algebraic structure is related to the fact that bosons with $\lambda = 0, 1, 2$ and 3 can be expressed as coupled fermions, each with $j = \frac{3}{2}$. In this respect, the algebraic lattice of U(16) that is detailed in this article is relevant to such bose–fermi models.

In such a large model, many subalgebras cannot be identified by their type, such as SU(3). For this reason, we adopt the notation of including a subscript indicating the types of bosons that are used to generate the algebra. For our purposes, this label (or the lack of a label) is a sufficiently unique classification. We note that the subalgebra structure of $U_{pf}(10)$ (Nadjakov and Mikhailov 1987), and the two decompositions $U_{spdf}(16) \supset SU(4) \oplus SU(4) \supset Sp(4) \oplus Sp(4) \supset O_{spdf}(4) \supset O_{pdf}(3)$ (Engel and Iachello 1987) and $U_{spdf}(16) \supset SU_{pdf}(15) \supset SU_{pdf}(3) \supset O_{pdf}(3)$ (Castaños *et al* 1986) have been introduced previously, although we will see that in the last limit the dynamical symmetry was not completely specified. The actual construction of the generators and Casimir invariants will be discussed elsewhere (Kusnezov 1989). The SU(4) $\oplus$ SU(4) $\supset$ SU(4) subalgebra has also been discussed in relation to odd–odd nuclei when the odd proton and odd neutron quasiparticles are restricted to $j = \frac{3}{2}$ configurations (Hübsch and Paar 1985, 1987, Hübsch *et al* 1985). As we shall see below, however, aside from this obvious SU(4) subalgebra, there is in fact another distinct embedding of SU(4) with completely different properties.

Clearly not all U(16) dynamical symmetry limits are relevant. Although there are 75 distinct subalgebras producing 165 dynamical symmetry limits, many of the dynamical symmetry limits correspond to decoupled excitations and identical subalgebras appear in many different symmetry breaking schemes. It is important to consider whether a few dynamical symmetry limits can be extracted that perhaps contain the essential dynamics of the model. A criterion by which such limits can be chosen is based on the inability of the sdf IBM to describe collective nuclear excitations. Thus, isolate the dynamical symmetry limits of $U_{spdf}(16)$ in which the p and f boson spaces never decouple as the $U_{spdf}(16)$ symmetry is broken down to $O_{pdf}(3)$. Further, we can require some algebraic coupling between positive and negative parity bosons at a level higher than $O_{pdf}(3)$. In this way we are led to a few limiting symmetries that are perhaps most physically relevant. These symmetry limits are referred to as pdf dynamical symmetries. Not all physically relevant symmetries fall into this category, since vibrational excitations are described by the decoupling of each boson into its separate harmonic oscillator algebra. However, in the study of octupole deformations, a strong interplay between the positive and negative parity bosons is needed, and is the basis of our selection. Further, when the physics indicates some decoupling, then one no longer requires the full U(16) algebra, and a smaller model is sufficient.

2. Maximal subalgebras of $U_{spdf}(16)$

The classification of the different maximal subalgebras is simplified by the techniques of Dynkin (Dynkin 1957, Cahn 1984). According to Dynkin, maximal subalgebras fall into two categories: regular subalgebras and non-regular (or S-) subalgebras. The difference between a regular and an S-subalgebra of some algebra G lies in the embedding of the root spaces of these subalgebras into the root space of the algebra G. Details can be found in standard references (Dynkin 1957, Gruber and Samuel 1968, Cahn 1984), and we shall not repeat them here. In terms of the interacting boson model, the difference between these subalgebras will now be discussed.

Table 1. Maximal simple and non-simple regular and S- subalgebras of the $U_{spdf}(16)$ interacting boson model which contain the physical angular momentum.

Model	Dynamical symmetry	Type of subalgebra	Type of embedding
$U_{spdf}(16)$		S-subalgebra	
	$O_{spdf}(16)$		simple
	$O_{spdf}(10)$		spinor, simple
	$SU_{spdf}(4) \oplus SU_{spdf}(4)$		non-simple
	$SU_{spdf}(2) \oplus SU_{spdf}(8)$		non-simple
		Regular subalgebra	
	$U_{pdf}(15)$		simple
	$U_p(3) \oplus U_{sdf}(13)$		non-simple
	$U_{sp}(4) \oplus U_{df}(12)$		non-simple
	$U_d(5) \oplus U_{spf}(11)$		non-simple
	$U_{sd}(6) \oplus U_{pf}(10)$		non-simple
	$U_f(7) \oplus U_{spd}(9)$		non-simple
	$U_{sf}(8) \oplus U_{pd}(8)$		non-simple

In determining the number of ways to break the U(16) symmetry of this model, we will require the condition that the one-boson representation of U(16) contains the O(3) representations $J = 0, 1, 2, 3$, corresponding to the four types of bosons. Then the number of maximal regular and S-subalgebras that obey this condition are precisely the number of ways to break the U(16) symmetry. Maximal regular subalgebras of a unitary algebra U(N) can be found from extended Dynkin diagrams (Dynkin 1957). Using the above condition on the angular momentum content of the subalgebras, it is not hard to see that the general form of the regular subalgebras of U(16) are $U(16) \supset U(16-n) \oplus U(n)$, where $n = \sum_{\{l_i\}\subset\{0,1,2,3\}}(2l_i + 1)$. Here l_i is summed over any number of the different boson spins. Regular subalgebras must be of this form since the one-boson 16-dimensional representation [1] of U(16), breaks into the representations $([1] \otimes [0]) \oplus ([0] \otimes [1])$ of $U(16-n) \oplus U(n)$, with the dimensions $(16 - n)$ and n, respectively. This sum of representations requires that the angular momentum be partitioned between U(n) and U(16$-n$). These are precisely the subalgebras formed by the partitioning of the four types of bosons into two groups. In this way the maximal simple and non-simple regular subalgebras of U(16) are easily identified, and given in table 1.

The general form of all non-simple maximal S-subalgebras has been tabulated by Dynkin (Dynkin 1957). For U(16), they are of the form $U(16) \supset U(n) \oplus U(m)$, where $n \times m = 16$. This results in the two choices $(n, m) = (4, 4)$ and $(2, 8)$. Both of these non-simple maximal subalgebras can be broken to $O_{pdf}(3)$ and hence have the proper

Table 2. Maximal subalgebras of unitary algebras that appear in the decomposition of U(16). These are the subalgebras formed by various groupings of the s, p, d and f bosons. Some of the shorter subalgebra chains are drawn to completion.

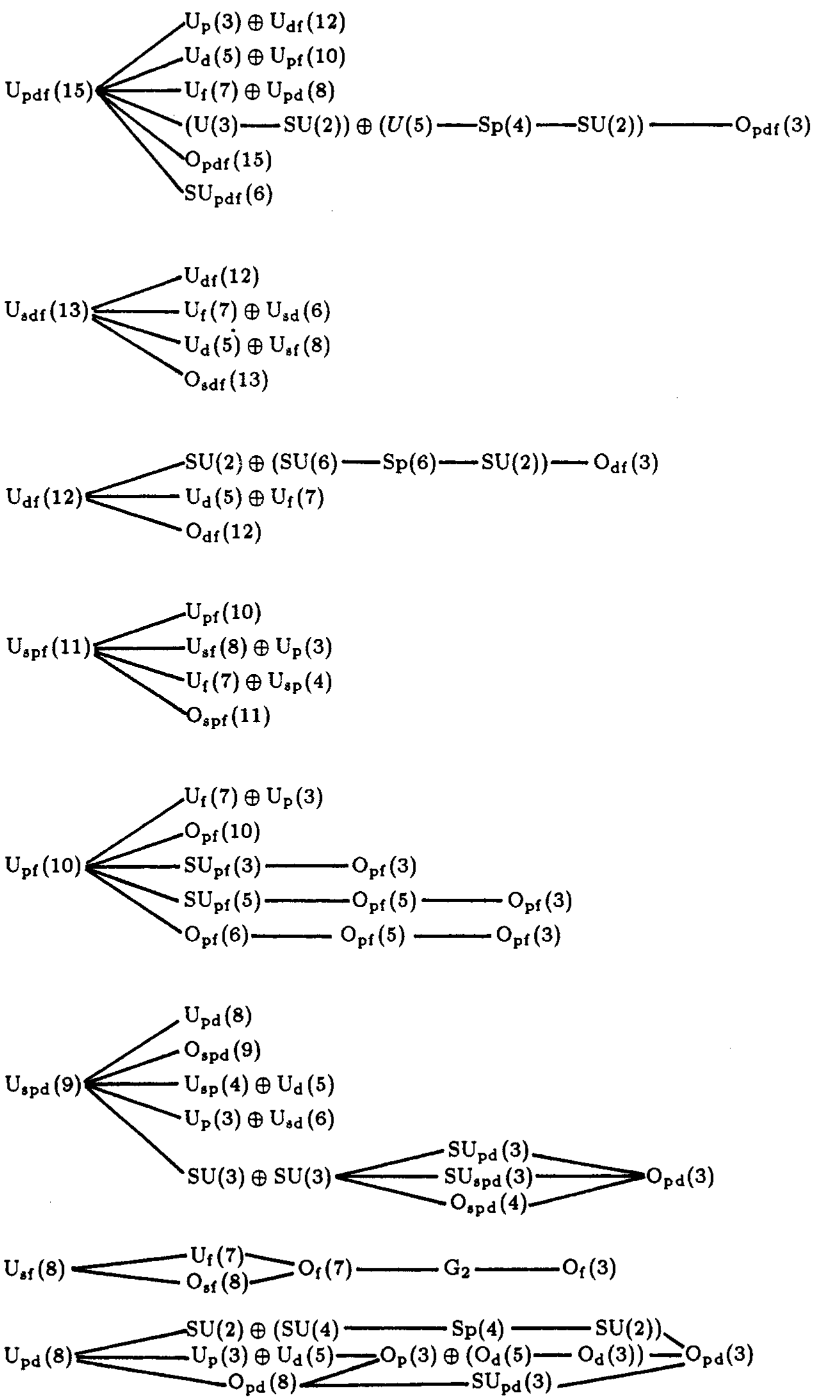
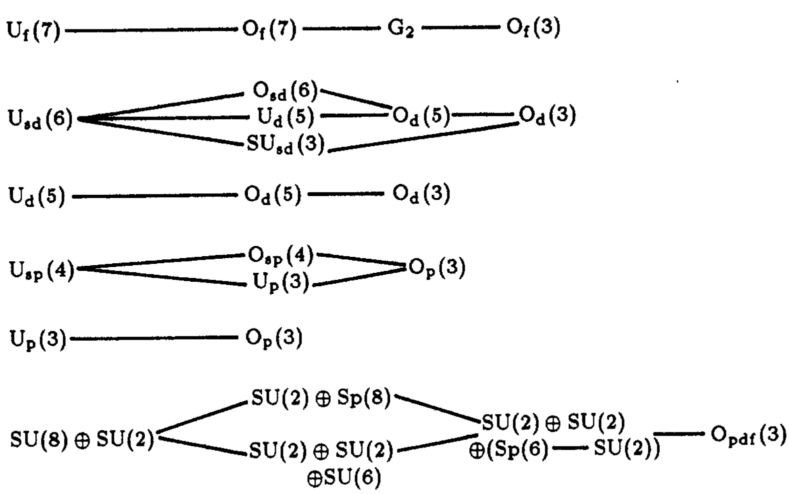


Table 2. (continued)



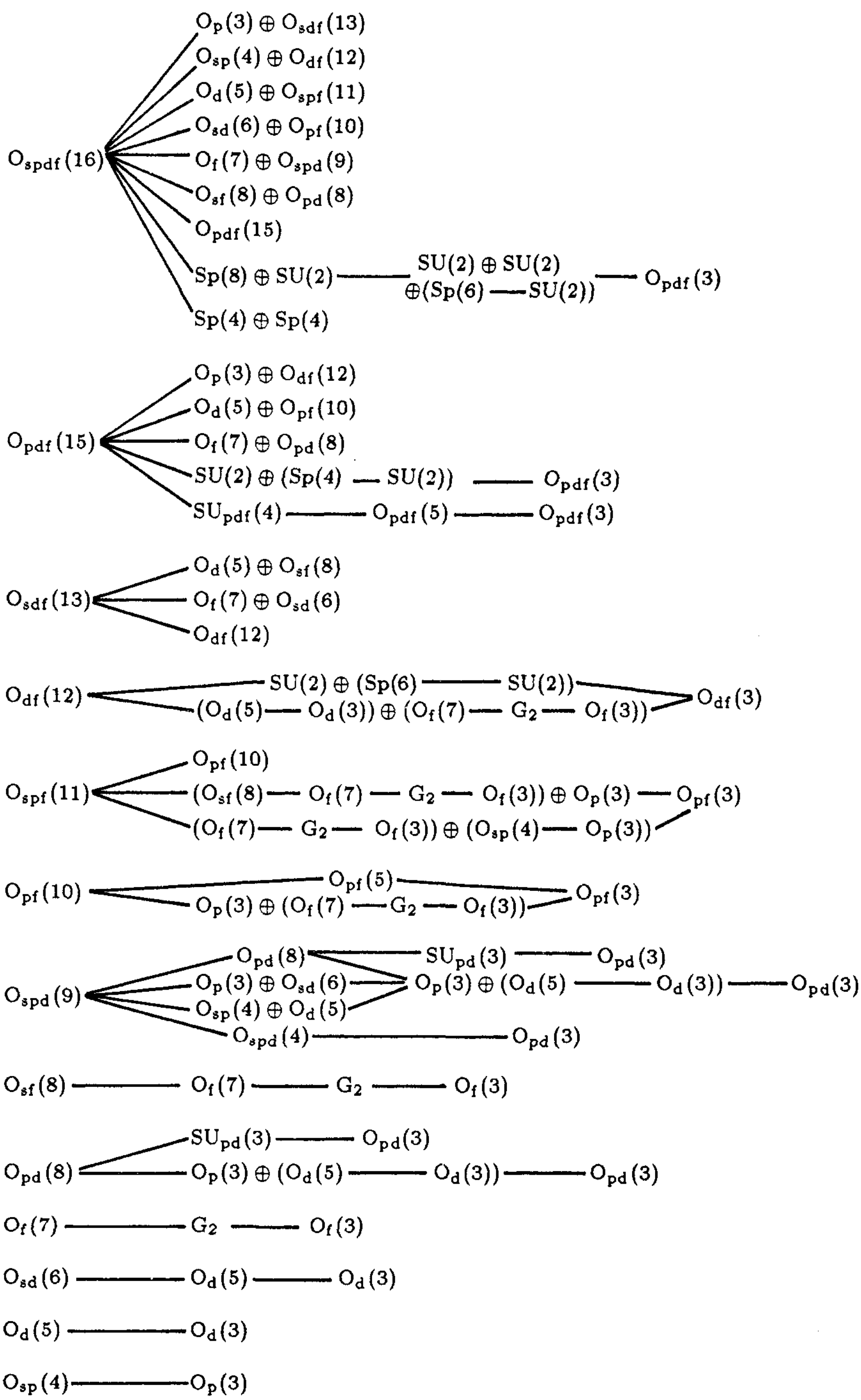
angular momentum content. The maximal simple S-subalgebras of U(16) can be found by identifying all algebras with lower rank that (i) have a 16-dimensional representation and no bilinear form, and (ii) are not among the few exceptions to this rule. This leads to two maximal simple S-subalgebras that can be embedded in [1] of U(16). They are the 16-dimensional symmetric representation of O(16), (10000000), and the 16-dimensional spinor representation of O(10), given by (00001). The maximal simple and non-simple regular and S-subalgebras of U(16) is given in table 1, indicating that there are eleven symmetry breaking schemes.

3. Classification of subalgebras

In order to unfold the full algebraic structure of this model, we must repeat this procedure for all subalgebras until we reach the physical angular momentum algebra generated by $O_{pdf}(3)$. The complete subalgebra lattice is too large to display in its entirety, and since most of it is not illuminating, we will instead list all maximal subalgebras and their maximal subalgebras and so forth. The full lattice can then easily be pieced together from this information. In tables 2 and 3 the maximal subalgebras of unitary and orthogonal algebras that appear in the decomposition of U(16) are tabulated. The cases where the symmetry can be broken to $O_{pdf}(3)$ in only a few steps are indicated in these tables. Several subalgebra decompositions that do not appear in tables 2 and 3 are discussed below and appear in table 4.

In table 4, there are two distinct SU(4) subalgebras of SU(4) $\oplus$ SU(4), which are referred to as $SU_{pdf}(4)$ and $O_{spdf}(6)$. Both these embeddings of SU(4) carry the same Dynkin index $j = 1$. Since the SU(4) $\oplus$ SU(4) subalgebra corresponds to the embedding of the (100) $\otimes$ (100) representation into [1] of $U_{spdf}(16)$, these algebras can be combined to form the $SU_{spdf}(4) \sim O_{spdf}(6)$ representations (200) $\oplus$ (010). However, there is also an embedding of the 15-dimensional and one-dimensional representations (101) $\oplus$ (000) of $SU_{pdf}(4)$. The essential difference between these algebras is that the latter does not

Table 3. Maximal subalgebras of orthogonal algebras that appear in the decomposition of U(16). Some of the simpler subalgebra chains are drawn to completion.



conserve the number of positive parity bosons, allowing mixing of positive and negative parity bosons. This is because $O_{spdf}(6)$ appears as a subalgebra of $U_{sd}(6) \oplus U_{pf}(10)$, which separately conserves the number of positive and negative parity bosons. Similarly, there are two distinct SU(3) embeddings in the SU(3) $\oplus$ SU(3) subalgebra of $U_{spd}(9)$, denoted $SU_{pd}(3)$ and $SU_{spd}(3)$. As with the SU(4) algebras, both SU(3) embeddings carry the same Dynkin index $j = 1$. Further, it was commented in Castaños *et al* (1986) that the 15-dimensional (2,1) representation of $SU_{pdf}(3)$ is a subalgebra of $SU_{pdf}(15)$. We remark that $SU_{pdf}(3)$ is actually a maximal subalgebra of $SU_{pdf}(6)$, which in turn is maximal in $SU_{pdf}(15)$. Thus, (2,1) can be embedded in the 15-dimensional (01000) representation of $SU_{pdf}(6)$, which can then be embedded in [1] of $SU_{pdf}(15)$. This leads to the chain $U_{spdf}(16) \supset SU_{pdf}(15) \supset SU_{pdf}(6) \supset SU_{pdf}(3) \supset O_{pdf}(3)$.

The difficulties encountered in the description of nuclear collective octupole excitations in the framework of the $U_{sdf}(13)$ algebra can be understood from tables 2 and 3. Aside from the Sp(6) $\oplus$ SU(2) limit of $O_{sdf}(12)$ (which does not contain the boson angular momentum operators as generators), the subalgebras of $U_{sdf}(13)$ mostly decouple the bosons into separate subalgebras, leading to decoupled excitations. Thus octupole deformed nuclei (reflection asymmetric shapes) do not arise naturally in any dynamical symmetry limit.

4. pdf dynamical symmetry limits

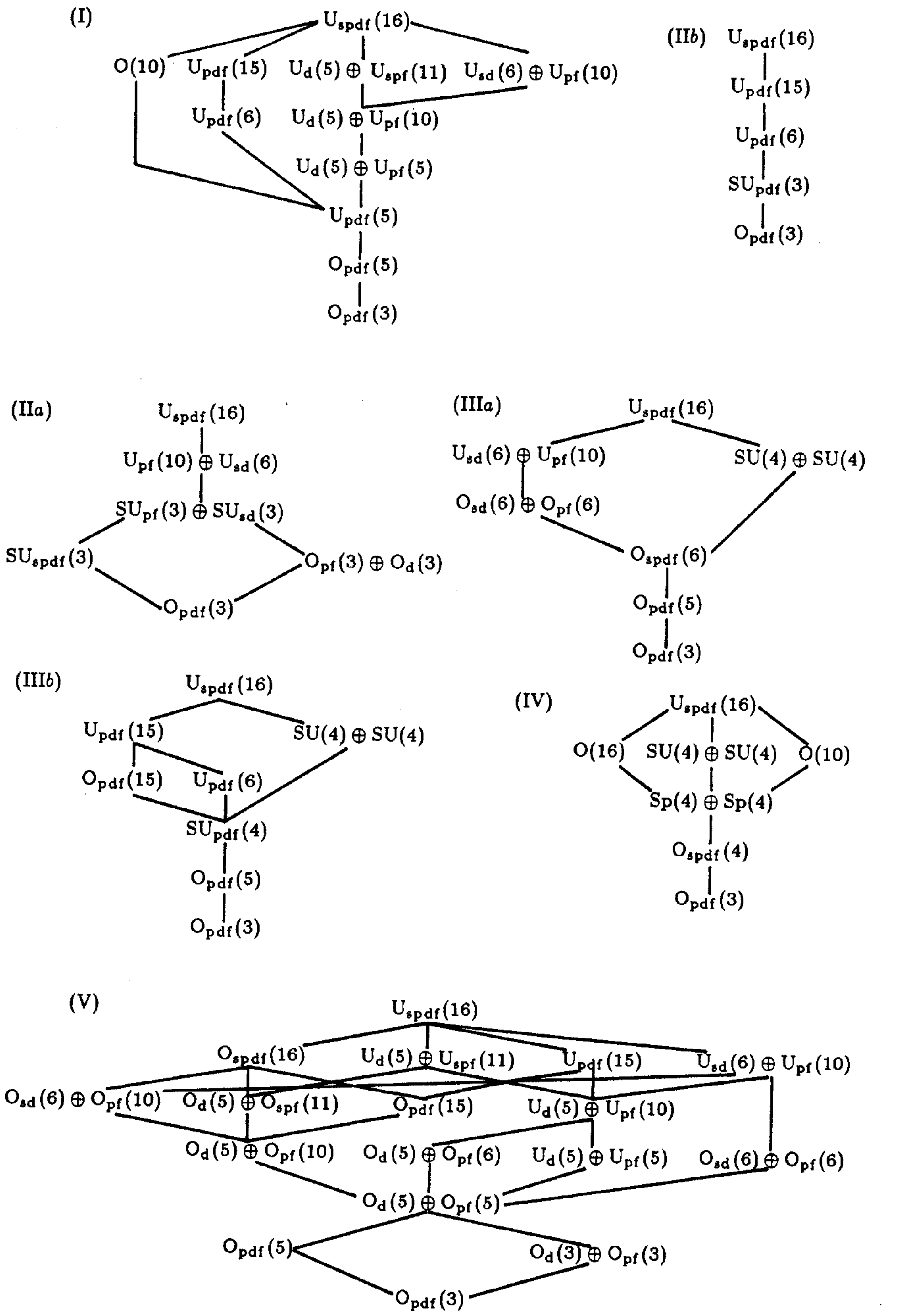
If we now consider the dynamical symmetries that do not decouple the p and f bosons and retain coupling to positive parity bosons at a level above $O_{pdf}(3)$, we find that there are seven dynamical symmetry limits with this property. They are the dynamical symmetry limits that contain the algebras $U_{pdf}(5)$, $O_{spdf}(6)$, $U_{pdf}(4)$, $SU_{spdf}(3)$, $SU_{pdf}(3)$, $O_{spdf}(4)$ and $O_{pf}(5) \oplus O_{d}(5)$. These special limits are shown in table 4. In order to preserve some of the notation of the sd IBM, we will number these limits (I), (II*a*), (II*b*), (III*a*), (III*b*), (IV) and (V). O(5) $\oplus$ O(5) has been taken as a limit rather than $O_{pdf}(5)$, since the latter appears in limits (I), (III*a*), (III*b*) and (IV), leading to one very large dynamical symmetry limit containing too many separate symmetries.

We have excluded the limits such as the $U_{pdf}(15) \supset$ SU(3) $\oplus$ SU(5), as well as the various symplectic limits involving SU(2) $\oplus$ Sp(6) and SU(2) $\oplus$ Sp(8). The reasoning is that the algebras in these limits couple only at the level of SU(2). They are also direct products of algebras, each of which contain angular momentum operators that are not physical boson angular momentum operators. When these limits are coupled at the SU(2) level, the angular momentum operator is finally recovered. So aside from the seven limits, all other dynamical symmetry limits either decouple p and f bosons, or only couple at the SU(2) level.

In studying these seven limits, it is useful to recall that the linear and quadratic Casimir invariants of the unitary algebras of form $U(\Sigma_{l_i}(2l_i+1))$ are related to $\hat{n}$ and $\hat{n}^2$, where $\hat{n}$ is the total number operator for the boson. The quadratic Casimir invariants of the orthogonal subalgebras of the type $O(\Sigma_{l_i}(2l_i + 1))$ are the pairing operators that can change the individual boson numbers by two, while conserving the overall boson number and parity. The Casimir invariants of the other algebras assume a more complex form (Kusnezov 1989), but nevertheless can be classified.

Using the results in table 4, the quadratic Casimir operators can be divided into three classes, which for convenience we refer to as A, B and C. Class A is defined as the generators (and hence Casimir invariants) of the algebras that appear as

Table 4. The pdf dynamical symmetry limits of $U_{spdf}(16)$.



subalgebras of $U_{sd}(6) \oplus U_{pf}(10)$ and necessarily separately conserve the number of negative and positive parity bosons. Hamiltonians constructed from the invariants of these subalgebras generate states of well defined parity, given by the expectation value of the number operator for negative parity bosons, $\langle \hat{N}_- \rangle$. Nearly all algebras in table 4, including the entire $SU_{spdf}(3)$ limit (II*a*), fall into this class. Naturally, parity doublets are not manifest in these limits since negative parity states can be moved with respect to positive parity states with the linear invariants of $U_{sd}(6)$ and $U_{pf}(10)$. The remaining algebras are the exceptions to this class. Class B are the algebras with Casimir operators that have good parity but do not commute with $\hat{N}_-$, while class C are the algebras with Casimir operators that are of mixed parity and hence do not commute with $\hat{N}_-$. There are only two algebras in table 4 of class C: $SU_{pdf}(6)$ and $SU_{pdf}(3)$. A method of constructing states of good parity for $SU_{pdf}(3)$ has been discussed in Castaños *et al* 1986, and can be used to classify the energy eigenstates of limit (II*b*) with respect to parity. The eight class B algebras are

$$\begin{array}{lll} SU(4) \oplus SU(4) & Sp(4) \oplus Sp(4) & \\ O_{spdf}(16) & O_{pdf}(15) & O_{spf}(11) \\ O(10) & O_{spdf}(4) & SU_{pdf}(4). \end{array}$$

Hamiltonians constructed from the invariants of these algebras have eigenstates of well defined parity. From this simple classification, it is clear that Hamiltonians that describe octupole deformation must include terms from class B or C. Only in this way can negative parity bosons be mixed into the ground-state wavefunction.

5. Concluding remarks

We have identified all subalgebras that occur in the decomposition of U(16). Using the tables, any dynamical symmetry limit of the $U_{spdf}(16)$ model can be constructed. In the study of collective nuclear properties, this algebra is relevant to not only even–even nuclei, but also to odd–odd nuclei described by two quasiparticles each in $j = \frac{3}{2}$ configurations. The current success of f and p bosons in describing collective nuclear properties of both positive and negative parity states in even–even nuclei suggests that the p boson should play an active role in the dynamics. Using this as a guide, we have identified seven dynamical symmetry limits in which there is no decoupling between the p and f bosons, and which retain coupling between positive and negative parity bosons at a level above $O_{pdf}(3)$. Further, we have classified the Casimir invariants of these limits with respect to parity. These pdf limits do not all have to reflect physical symmetries. However, if the presence of p and f bosons are of equal importance in the nuclear structure of low-lying negative parity states, some of these symmetries should be realised in some collective nuclei. These limits provide a starting point for dynamical symmetry studies involving the $U_{spdf}(16)$ model.

Acknowledgments

The author would like to thank F Iachello for his encouragement and continued interest in this problem. Support for this work was provided by the National Science Foundation under Grant No. 87-14432.

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